Topic: Using trig identities to prove the equation

Question: Which of the following is an identity?

Answer choices:

A $\dfrac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \sin(2\theta)$

B $\dfrac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \tan(2\theta)$

C $\dfrac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \cos(2\theta)$

D $\dfrac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \cot(2\theta)$

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Solution: B

First, we'll work on the numerator of the expression on the left-hand side of the equation in answer choice B.

By one of the sum-to-product identities,

$$\sin\theta + \sin(3\theta) = 2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)$$

$$\sin\theta + \sin(3\theta) = 2\sin\left(\frac{4\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$

$$\sin\theta + \sin(3\theta) = 2\sin(2\theta)\cos(-\theta)$$

Therefore,

$$\frac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \frac{2\sin(2\theta)\cos(-\theta)}{\cos\theta + \cos(3\theta)}$$

Next, we'll work on the denominator.

By another sum-to-product identity,

$$\cos\theta + \cos(3\theta) = 2\cos\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)$$

$$\cos\theta + \cos(3\theta) = 2\cos\left(\frac{4\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$

$$\cos\theta + \cos(3\theta) = 2\cos(2\theta)\cos(-\theta)$$

Substituting this result, we obtain

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$$\frac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \frac{2\sin(2\theta)\cos(-\theta)}{2\cos(2\theta)\cos(-\theta)}$$

After doing all the cancellation that's possible, we get

$$\frac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \frac{\sin(2\theta)}{\cos(2\theta)}$$

By the definition of the tangent (as the ratio of the sine to the cosine), we obtain the expression on the right-hand side of the equation in answer choice B:

$$\frac{\sin\theta + \sin(3\theta)}{\cos\theta + \cos(3\theta)} = \tan(2\theta)$$

To see that none of the other answer choices is an identity, note that the left-hand side of all four answer choices is the same, so it suffices to exhibit some angle θ for which the right-hand sides of all of them are different.

Consider $\theta = \pi/6$. For answer choice B (the equation we have shown to be an identity), we obtain

$$\tan(2\theta) = \tan\left[2\left(\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$$

For answer choice A, we get

$$\sin(2\theta) = \sin\left[2\left(\frac{\pi}{6}\right)\right] = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \neq \sqrt{3}$$

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For answer choice C,

$$\cos(2\theta) = \cos\left[2\left(\frac{\pi}{6}\right)\right] = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \neq \sqrt{3}$$

For answer choice D,

$$\cot(2\theta) = \frac{1}{\tan(2\theta)} = \frac{1}{\sqrt{3}} \neq \sqrt{3}$$

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Topic: Using trig identities to prove the equation

Question: Which of the following is not an identity?

Answer choices:

A $\dfrac{\sec\theta \sec\alpha}{\sec(\theta - \alpha)} = 1 + \tan\theta \tan\alpha$

B $\sin(4\theta) = 2\sin(2\theta)(1 - 2\sin^2\theta)$

C $(\sin\theta)\left(\tan\dfrac{\theta}{2}\right) = 1 + \cos\theta$

D $\dfrac{\cos(2\theta) - \cos(4\theta)}{\cos(2\theta) + \cos(4\theta)} = \tan(3\theta)(\tan\theta)$

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Solution: C

To see that the equation in answer choice C is not an identity, let $\theta = 0$. Then

$$(\sin\theta)\left(\tan\frac{\theta}{2}\right) = (\sin 0)\left(\tan\frac{0}{2}\right) = 0(0) = 0$$

but

$$1 + \cos\theta = 1 + \cos(0) = 1 + 1 = 2 \neq 0$$

We will prove that each of the other answer choices is indeed an identity.

Answer choice A is

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = 1 + \tan\theta\tan\alpha$$

To prove this, we'll first use the reciprocal identity for secant (i.e., that the secant is the reciprocal of the cosine), applied to all three occurrences of "sec":

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = \frac{\left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\cos\alpha}\right)}{\left[\frac{1}{\cos(\theta-\alpha)}\right]}$$

Combining the two fractions in the numerator, we have

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = \frac{\left(\frac{1}{\cos\theta\cos\alpha}\right)}{\left[\frac{1}{\cos(\theta-\alpha)}\right]}$$

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Next, we'll invert the overall fraction:

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = \left(\frac{1}{\cos\theta\cos\alpha}\right)\left[\frac{\cos(\theta-\alpha)}{1}\right]$$

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = \frac{\cos(\theta-\alpha)}{\cos\theta\cos\alpha}$$

By the difference identity for cosine,

$$\cos(\theta-\alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

Therefore,

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = \frac{\cos\theta\cos\alpha + \sin\theta\sin\alpha}{\cos\theta\cos\alpha}$$

Expressing the fraction as the sum of two fractions, we get

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = \frac{\cos\theta\cos\alpha}{\cos\theta\cos\alpha} + \frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}$$

Simplifying the first fraction, we obtain

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = 1 + \frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}$$

Expressing the remaining fraction as a product of two fractions, we have

$$\frac{\sec\theta\sec\alpha}{\sec(\theta-\alpha)} = 1 + \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin\alpha}{\cos\alpha}\right)$$

Using the definition of the tangent (as the ratio of the sine to the cosine), we get the desired result:

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$$\frac{\sec\theta \sec\alpha}{\sec(\theta - \alpha)} = 1 + \tan\theta \tan\alpha$$

Answer choice B is

$$\sin(4\theta) = 2\sin(2\theta)(1 - 2\sin^2\theta)$$

Thinking of 4θ as $2(2\theta)$ and applying the double-angle identity for sine to the angle 2θ, we get

$$\sin(4\theta) = \sin(2(2\theta)) = 2\sin(2\theta)\cos(2\theta)$$

By the double-angle identity for cosine,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Thus we obtain

$$\sin(4\theta) = 2\sin(2\theta)(\cos^2\theta - \sin^2\theta)$$

By the basic Pythagorean identity,

$$\cos^2\theta = 1 - \sin^2\theta$$

Therefore,

$$\sin(4\theta) = 2\sin(2\theta)\left[(1 - \sin^2\theta) - \sin^2\theta\right]$$

Regrouping:

$$\sin(4\theta) = 2\sin(2\theta)\left[1 + (-\sin^2\theta - \sin^2\theta)\right]$$

Simplifying this, we get the desired result:

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$$\sin(4\theta) = 2\sin(2\theta)(1 - 2\sin^2\theta)$$

Answer choice D is

$$\frac{\cos(2\theta) - \cos(4\theta)}{\cos(2\theta) + \cos(4\theta)} = \tan(3\theta)(\tan\theta)$$

First, we'll work on the numerator of the expression on the left-hand side of the equation.

By one of the sum-to-product identities,

$$\cos(2\theta) - \cos(4\theta) = -2\sin\left(\frac{2\theta + 4\theta}{2}\right)\sin\left(\frac{2\theta - 4\theta}{2}\right)$$

$$\cos(2\theta) - \cos(4\theta) = -2\sin\left(\frac{6\theta}{2}\right)\sin\left(\frac{-2\theta}{2}\right)$$

$$\cos(2\theta) - \cos(4\theta) = -2\sin(3\theta)\sin(-\theta)$$

By the odd identity for sine (i.e., that $\sin(-\theta) = -\sin\theta$), we have

$$\cos(2\theta) - \cos(4\theta) = -2\sin(3\theta)(-\sin\theta) = 2\sin(3\theta)(\sin\theta)$$

Substituting this result for the numerator in our expression, we get

$$\frac{\cos(2\theta) - \cos(4\theta)}{\cos(2\theta) + \cos(4\theta)} = \frac{2\sin(3\theta)(\sin\theta)}{\cos(2\theta) + \cos(4\theta)}$$

Next, we'll work on the denominator.

By another sum-to-product identity,

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$$\cos(2\theta)+\cos(4\theta)=2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right)$$

$$\cos(2\theta)+\cos(4\theta)=2\cos\left(\frac{6\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$

$$\cos(2\theta)+\cos(4\theta)=2\cos(3\theta)\cos(-\theta)$$

By the even identity for cosine (i.e., that $\cos(-\theta)=\cos\theta$), we have

$$\cos(2\theta)+\cos(4\theta)=2\cos(3\theta)(\cos\theta)$$

Substituting this result gives

$$\frac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)}=\frac{2\sin(3\theta)(\sin\theta)}{2\cos(3\theta)(\cos\theta)}$$

Cancellation of the 2's gives

$$\frac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)}=\frac{\sin(3\theta)(\sin\theta)}{\cos(3\theta)(\cos\theta)}$$

Expressing the fraction as a product of two fractions, we find that

$$\frac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)}=\left[\frac{\sin(3\theta)}{\cos(3\theta)}\right]\left(\frac{\sin\theta}{\cos\theta}\right)$$

By the definition of the tangent (as the ratio of the sine to the cosine), we get the desired result:

$$\frac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)}=\tan(3\theta)(\tan\theta)$$

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Topic: Using trig identities to prove the equation

Question: Which of the following is an identity?

Answer choices:

A $$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta+\sin\left(\frac{\theta}{2}\right)\right]$$

B $$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta-\sin\left(\frac{\theta}{2}\right)\right]$$

C $$\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\cos\theta-\cos\left(\frac{\theta}{2}\right)\right]$$

D $$\sin\left(\frac{\theta}{4}\right)\sin\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\cos\theta+\cos\left(\frac{\theta}{2}\right)\right]$$

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Solution: B

First, we'll prove that the equation given in answer choice B is an identity:

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta-\sin\left(\frac{\theta}{2}\right)\right]$$

Well, by one of the product-to-sum identities,

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\left(\frac{\theta}{4}+\frac{3\theta}{4}\right)+\sin\left(\frac{\theta}{4}-\frac{3\theta}{4}\right)\right]$$

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\left(\frac{\theta+3\theta}{4}\right)+\sin\left(\frac{\theta-3\theta}{4}\right)\right]$$

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\left(\frac{4\theta}{4}\right)+\sin\left(\frac{-2\theta}{4}\right)\right]$$

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta+\sin\left(-\frac{\theta}{2}\right)\right]$$

By the odd identity for sine (i.e., that $\sin(-\theta/2)=-\sin(\theta/2)$), we have

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta-\sin\left(\frac{\theta}{2}\right)\right]$$

Now we'll show that none of the other three answer choices is an identity.

Answer choice A is

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$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta+\sin\left(\frac{\theta}{2}\right)\right]$$

We'll show that this is false for $\theta=\pi$:

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)=\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}$$

but

$$\frac{1}{2}\left[\sin\theta+\sin\left(\frac{\theta}{2}\right)\right]=\frac{1}{2}\left[\sin\pi+\sin\left(\frac{\pi}{2}\right)\right]=\frac{1}{2}(0+1)=\frac{1}{2}(1)=\frac{1}{2}\neq-\frac{1}{2}$$

Answer choice C is

$$\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\cos\theta-\cos\left(\frac{\theta}{2}\right)\right]$$

We'll show that this is false for $\theta=2\pi$:

$$\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)$$

$$\cos\left(\frac{2\pi}{4}\right)\cos\left[\frac{3(2\pi)}{4}\right]$$

$$\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{6\pi}{4}\right)$$

$$\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right)$$

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$$0(0) = 0$$

but

$$\frac{1}{2}\left[\cos\theta - \cos\left(\frac{\theta}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) - \cos\left(\frac{2\pi}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) - \cos(\pi)\right]$$

$$\frac{1}{2}[1 - (-1)]$$

$$\frac{1}{2}(2) = 1 \neq 0$$

Answer choice D is

$$\sin\left(\frac{\theta}{4}\right)\sin\left(\frac{3\theta}{4}\right) = \frac{1}{2}\left[\cos\theta + \cos\left(\frac{\theta}{2}\right)\right]$$

We'll show that this is false for $\theta = 2\pi$:

$$\sin\left(\frac{\theta}{4}\right)\sin\left(\frac{3\theta}{4}\right)$$

$$\sin\left(\frac{2\pi}{4}\right)\sin\left[\frac{3(2\pi)}{4}\right]$$

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$$\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{6\pi}{4}\right)$$

$$\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)$$

$$1(-1) = -1$$

but

$$\frac{1}{2}\left[\cos\theta + \cos\left(\frac{\theta}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) + \cos\left(\frac{2\pi}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) + \cos(\pi)\right]$$

$$\frac{1}{2}[1 + (-1)]$$

$$\frac{1}{2}(0) = 0 \neq -1$$

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